

Generalized β -Gaussian Ensemble Equilibrium measure method

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Abstract

We describe β -Generalized random Hermitian matrices ensemble sometimes called Chiral ensemble. We give global asymptotic of the density of eigenvalues or the statistical density. We investigate general method names as equilibrium measure method. When taking n large limit we will see that the asymptotic density of eigenvalues generalize the Wigner semi-circle law.

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1 Introduction

The generalized β -Gaussian ensemble, generalize the classical random matrix ensemble: Gaussian orthogonal, unitary and symplectic ensembles (denoted by GOE, GUE and GSE for short, which correspond to the Dyson index $\beta = 1, 2$ and 4), from the quantization index to the continuous exponents $\beta > 0$. These ensembles possess the joint probability density function (p.d.f.) of real eigenvalues $\lambda_1, \dots, \lambda_n$ with the form

$$\mathbb{P}_n(d\lambda) = \frac{1}{Z_n} e^{-\sum_{i=1}^n \lambda_i^2} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta d\lambda_1 \dots d\lambda_n,$$

where Z_n can be evaluated by the using the Selberg integral

$$Z_n = (2\pi)^{2n} \prod_{i=1}^n \frac{\Gamma(1 + \frac{\beta_i}{2})}{\Gamma(1 + \frac{\beta}{2})}.$$

Recently, Dumitriu and Eldeman have construct a tri-diagonal matrix model of these ensembles see [3].

Basing on the p.d.f. of eigenvalues \mathbb{P}_n , the (level) density, or one-dimensional marginal eigenvalue density scaled by the factor $\frac{1}{\sqrt{2n}}$ converge weakly to the famous Wigner semi-circle law as follows: for every bounded continuous functions f on \mathbb{R}

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f\left(\frac{x}{\sqrt{2n}}\right) h_n(x) dx = \int_{\mathbb{R}} f(x) \rho(x) dx,$$

where

$$\rho(x) = \begin{cases} \frac{1}{\pi} \sqrt{2 - x^2} & \text{if } |x| \leq \sqrt{2} \\ 0 & \text{if } |x| \geq \sqrt{2} \end{cases}$$

$$h_n(\lambda_1) = \int_{\mathbb{R}^{n-1}} \mathbb{P}_n(\lambda_1, \dots, \lambda_n) d\lambda_2 \dots d\lambda_n.$$

Many others work in this direction of random matrices and asymptotic of eigenvalues has been developed in the last years, one can see [6], for a good reference.

In this work we will study a generalization of the Gaussian random matrices ensemble which is called some times the Chiral-ensemble when $\beta = 1, 2$ or 4 . We will consider the general case where $\beta > 0$, in that case the joint probability density in \mathbb{R}^n is given by:

$$\mathbb{P}_n(dx) = \frac{1}{Z_n} e^{-\sum_{i=1}^n x_i^2} \prod_{i=1}^n |x_i|^{2\lambda} \prod_{1 \leq i < j \leq n} |x_i - x_j|^\beta dx_1 \dots dx_n,$$

where Z_n is a normalizing constant and λ is a positive parameter. Using a general method of logarithmic potential we will prove that, the statistical density of eigenvalues converge for the tight topology as $n \rightarrow +\infty$ to some probability density. Which generalized the Wigner semi-circle law. Such result has been proved in [1] for $\beta = 2$, by the orthogonal polynomials method.

The paper is organized as follow. In sections 2 and 3 we gives some results about classical potential theory, which will be used together with some fact about boundary values distribution to characterized the Cauchy transform of some equilibrium measures.

In section 4, we will describe the model to study, as physics model, and we give the joint probability density. Moreover we defined the statistical density ν_n of eigenvalues and we explain how the eigenvalues must be rescaling by the factor \sqrt{n} . Also we gives the first means result theorem 4.1, which state the convergence of the statistical density ν_n to some probability measure $\nu_{\beta,c}$. We will prove that, the measure $\nu_{\beta,c}$ is an equilibrium measure and we compute the exact value of the energy for general β , after calculating the energy for $\beta = 2$.

In section 5 we gives the proof of the first result of theorem 4.1.

2 Logarithmic potential

The logarithmic potential of a positive measure ν on \mathbb{R} is the function U^ν defined by

$$U^\nu(x) = \int_{\mathbb{R}} \log \frac{1}{|x-t|} \nu(dt).$$

It will defined with value on $] -\infty, +\infty]$ if ν is with compactly support or more general, if

$$\int_{\mathbb{R}} \log(1+|t|) \nu(dt) < \infty.$$

Observe that

$$\lim_{n \rightarrow \infty} (U^\nu(x) + \nu(\mathbb{R}) \log |x|) = 0.$$

The Cauchy transform G_ν of a bounded measure ν on \mathbb{R} is the function defined on $\mathbb{C} \setminus \text{supp}(\nu)$ by

$$G_\nu(z) = \int_{\mathbb{R}} \frac{1}{z-t} \nu(dt).$$

The Cauchy transform is holomorphic.

Assume that $\text{supp}(\nu) \subset] -\infty, a]$, and

$$\int_{\mathbb{R}} \log(1+|t|) \nu(dt) < \infty.$$

Then the function

$$F(z) = \int_{\mathbb{R}} \log \frac{1}{|z-t|} \nu(dt).$$

is defined and holomorphic in $\mathbb{C} \setminus]-\infty, a]$. Furthermore $F'(z) = G_\nu(z)$, and

$$\begin{aligned} U^\nu(x) &= -\operatorname{Re} F(x) & (x > a) \\ U^\nu(x) &= -\lim_{\varepsilon \rightarrow 0} \operatorname{Re} F(x + i\varepsilon) & (x \in \mathbb{R}) \end{aligned}$$

In the distribution sense,

$$\frac{d}{dx} U^\nu(x) = -\operatorname{Re} G_\nu(x).$$

We will use some properties of the boundary value distribution of a holomorphic function. Let f be holomorphic in $\mathbb{C} \setminus \mathbb{R}$. It is said to be of moderate growth near \mathbb{R} if, for every compact set $K \subset \mathbb{R}$, there are $\varepsilon > 0$, $N > 0$, and $C > 0$ such that

$$|f(x + iy)| \leq \frac{C}{|y|^N} \quad (x \in K, 0 < |y| \leq \varepsilon).$$

Then for all $\varphi \in \mathcal{D}(\mathbb{R})$,

$$(T, \varphi) = \lim_{\varepsilon \rightarrow 0, \varepsilon > 0} \int_{\mathbb{R}} \varphi(x) (f(x + i\varepsilon) - f(x - i\varepsilon)) dx,$$

defines a distribution T on \mathbb{R} . It is denoted $T = [f]$, and called the difference of boundary values of f . One shows that the function f extends as a holomorphic function in $\mathbb{C} \setminus \operatorname{supp}([f])$. In particular, if $[f] = 0$, then f extends as a holomorphic function in \mathbb{C} .

For $\alpha \in \mathbb{C}$, the distribution Y_α is defined, for $\operatorname{Re} \alpha > 0$, by

$$(Y_\alpha, \varphi) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} \varphi(t) t^{\alpha-1} dt.$$

The distribution Y_α , as a function of α , admits an analytic continuation for $\alpha \in \mathbb{C}$. In particular $Y_0 = \delta_0$, the Dirac measure at 0.

For $\alpha \in \mathbb{C}$, we defines the holomorphic function z^α in $\mathbb{C} \setminus]-\infty, 0]$ as follows: if $z = re^{i\theta}$, with $r > 0$, $-\pi < \theta < \pi$, then

$$z^\alpha = r^\alpha e^{i\alpha\theta}.$$

The function z^α is of moderate growth near \mathbb{R} , and

$$([z^\alpha], \varphi) = -2i\pi \frac{1}{\Gamma(-\alpha)} (Y_{\alpha+1}, \check{\varphi}),$$

where $\check{\varphi}(t) = \varphi(-t)$. In particular when $\alpha = -1$

$$[\frac{1}{z}] = -2i\pi\delta_0.$$

Proposition 2.1 *Let ν be a bounded positive measure on \mathbb{R} .*

(i) *The Cauchy transform G_ν of ν is holomorphic in $\mathbb{C} \setminus \text{supp}(\nu)$, of moderate growth near \mathbb{R} , and*

$$[G_\nu] = -2i\pi\nu.$$

(ii) *Assume that the support of ν is compact. Let F be holomorphic in $\mathbb{C} \setminus \mathbb{R}$, of moderate growth near \mathbb{R} , such that*

$$[F] = -2i\pi\nu.$$

Then F is holomorphic in $\mathbb{C} \setminus \text{supp}(\nu)$. If further

$$\lim_{|z| \rightarrow \infty} F(z) = 0.$$

Then

$$G_\nu = F.$$

3 Equilibrium measure some basic results

Let us first recall some basic facts about the tight topology. All the present result in equilibrium measure can be find in the good reference [11] and references therein. Let $\mathfrak{M}^1(\Sigma)$ be the set of probability measures on the closed set $\Sigma \subset \mathbb{R}$. We consider the tight topology. For this topology a sequence (ν_n) converges to a measure ν if, for every continuous bounded function f on Σ ,

$$\lim_{n \rightarrow \infty} \int_{\Sigma} f(x) \nu_n(dx) = \int_{\Sigma} f(x) \nu(dx).$$

This topology is metrizable. If Σ is bounded, then $\mathfrak{M}^1(\Sigma)$ is compact.

Let Σ be a closed interval ($\Sigma = \mathbb{R},]-\infty, a], [b, +\infty[$ or $[a, b]$), and Q a function defined on Σ with values on $]-\infty, +\infty]$, continuous on $\text{int}(\Sigma)$. If Σ is unbounded, it is assumed that

$$\lim_{|x| \rightarrow +\infty} (Q(x) - \log(1 + x^2)) = +\infty.$$

If ν is a probability measure supported by Σ , the energy $E(\nu)$ of ν is defined by

$$E(\nu) = \int_{\Sigma \times \Sigma} \log \frac{1}{|x - y|} \nu(dx) \nu(dy) + \int_{\Sigma} Q(x) \nu(dx).$$

which mean that

$$E(\nu) = \int_{\Sigma} U^{\nu}(x) \nu(dx) + \int_{\Sigma} Q(x) \nu(dx).$$

By a straightforward computation we can prove that $E(\nu)$ is bounded below. Hence we defined

$$E^* = \inf\{E(\nu) \mid \nu \in \mathfrak{M}^1(\Sigma)\}.$$

Theorem 3.1 *If $\nu(dx) = f(x)dx$, where f is a continuous function with compact support $\subset \text{int}(\Sigma)$. Then the potential U^{ν} is a continuous function, and $E^* \leq E(\nu) < \infty$. Furthermore there is a unique measure $\nu^* \in \mathfrak{M}^1(\Sigma)$ such that*

$$E^* = E(\nu^*).$$

The support of ν^ is compact.*

This measure ν^ is called the equilibrium measure.*

Proposition 3.2 *Let $\nu \in \mathfrak{M}^1(\Sigma)$ with compact support. Assume that the potential U^{ν} of ν is continuous and that there is a constant C such that*

- (i) $U^{\nu}(x) + \frac{1}{2}Q(x) \geq C$ on Σ .
- (ii) $U^{\nu}(x) + \frac{1}{2}Q(x) = C$ on $\text{supp}(\nu)$. Then ν is the equilibrium measure: $\nu = \nu^*$.

The constant C is called the (modified) Robin constant. Observe that

$$E^* = C + \frac{1}{2} \int_{\Sigma} Q(x) \nu^*(dx).$$

It is easy to see the action by linear transformation on the energy.

Proposition 3.3 *Let the transformation $h(s) = as + b$ map Σ onto Σ' . If Q is defined on Σ' , then $Q \circ h$ is defined on Σ . If ν is a probability measure on Σ , then $\sigma = h(\nu)$ is the probability measure on Σ' defined by*

$$\int_{\Sigma'} f(t) \sigma(dt) = \int_{\Sigma} f \circ h(t) \nu(dt).$$

Then

$$E_{(\Sigma', Q)}(h(\nu)) = E_{(\Sigma, Q \circ h)}(\nu) - \log |a|.$$

For the proof of the previous theorem and proposition, see for instance theorem II.2.3, proposition II.3.1 of [4].

4 Statistical of the generalized Gaussian unitary ensemble

Let $H_n = \text{Herm}(n, \mathbb{F})$ be the vector space of square Hermitian matrices with coefficient in the field $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} . For $\mu > -\frac{1}{2}$, we denote by $\mathbb{P}_{n, \mu}$ the probability measure on H_n defined by.

$$\int_{H_n} f(x) \mathbb{P}_{n, \mu}(dx) = \frac{1}{C_n} \int_{H_n} f(x) |\det(x)|^{2\mu} e^{-\text{tr}(x^2)} m_n(dx),$$

for a bounded measurable function f , where m_n is the Euclidean measure associated to the usual inner product $\langle x, y \rangle = \text{tr}(xy)$ on H_n and C_n is a normalized constant. which is given for $d = 2$ by

$$C_n = n! \prod_{k=0}^{n-1} \gamma_{\mu}(k)$$

$$\gamma_{\mu}(k) = \begin{cases} m! \Gamma(m + \mu + \frac{1}{2}) & \text{if } k = 2m, \\ m! \Gamma(m + \mu + \frac{3}{2}) & \text{if } k = 2m + 1. \end{cases} \quad (4.1)$$

For general $\beta = 1$ or 4 the constant is given by Jack polynomials.

When $\mu = 0$ we recover the classical Gaussian unitary ensemble and,

$$C_n = \pi^{\frac{n}{2}} 2^{\frac{n(n-1)}{2}} \prod_{k=0}^{n-1} k!.$$

We endowed the space H_n with the probability measure $\mathbb{P}_{n,\mu}$. The probability $\mathbb{P}_{n,\mu}$ is invariant for the action of the unitary group $U(n)$ by the conjugation

$$x \mapsto uxu^* \quad (u \in U(n)).$$

4.1 Spectral density of eigenvalues

Let f be a $U(n)$ -invariant function on H_n .

$$f(uxu^*) = f(x) \quad \forall u \in U(n),$$

Then by the spectral theorem there exist a symmetric function F in \mathbb{R}^n such that

$$f(x) = F(\lambda_1, \dots, \lambda_n).$$

If f is integrable with respect to $\mathbb{P}_{n,\mu}$, then by using the formula of integration of Well we obtain

$$\int_{H_n} f(x) \mathbb{P}_{n,\mu}(dx) = \int_n F(\lambda_1, \dots, \lambda_n) q_{n,\mu}(\lambda_1, \dots, \lambda_n) d\lambda_1 \dots \lambda_n,$$

where

$$q_{n,\mu}(\lambda_1, \dots, \lambda_n) = \frac{1}{c_n} e^{-\sum_{k=1}^n \lambda_k^2} \prod_{k=1}^n |\lambda_k|^{2\mu} |\Delta(\lambda)|^\beta,$$

and $\beta = 1, 2, 4$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} . Δ is the vandermonde determinant

$$\Delta(\lambda) = \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j),$$

.

More general we will consider n particles free to move in \mathbb{R}^n , in equilibrium at absolute temperature T . A fundamental postulate gives the p.d.f. for the event that the particles are at positions $\lambda_1, \dots, \lambda_n$ as:

$$q_{n,\mu_n}(\lambda_1, \dots, \lambda_n) = \frac{1}{Z_N} e^{-\beta V_n(\lambda_1, \dots, \lambda_n)},$$

where

$$V_n(\lambda_1, \dots, \lambda_n) = \frac{2n}{\beta} \sum_{k=1}^n (\lambda_k^2 + \frac{2\mu_n}{n} \log \frac{1}{|\lambda_k|}) + \sum_{1 \leq i < j \leq n} \log \frac{1}{|\lambda_i - \lambda_j|}.$$

Here $V_n(\lambda_1, \dots, \lambda_n)$ denotes the total potential energy of the system, $\beta := \frac{1}{k_B T}$ (k_B is Boltzmann's constant), and Z_n is a normalizing constant.

The term $V_n(\lambda_1, \dots, \lambda_n)$ is referred to as the Boltzmann factor and $\tilde{Z}_n := \frac{Z_n}{n!}$ is called the (canonical) partition function.

Our first result is to study as n goes to infinity the asymptotic of the Normalized Counting Measure (Density of States) ν_n defined on \mathbb{R} as follows: if f is a measurable function,

$$\int_{\mathbb{R}} f(t) \nu_n(dt) = \mathbb{E}_{n, \mu_n} \left(\frac{1}{n} \sum_{i=1}^n f(\lambda_i) \right),$$

where \mathbb{E}_{n, μ_n} is the expectation with respect to the probability measure on \mathbb{R}^n

$$\mathbb{P}_{n, \mu_n}(d\lambda) = \frac{1}{Z_n} e^{-n \sum_{k=1}^n Q_n(\lambda_k)} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta,$$

and

$$Q_n(x) = x^2 + \frac{2\mu_n}{n} \log \frac{1}{|x|}.$$

By invariance of the measure \mathbb{P}_{n, μ_n} by the symmetric group, we have that the measure ν_n is continuous with respect to the Lebesgue measure

$$\nu_n(dt) = h_{n, \mu_n}(t) dt.$$

where

$$h_{n, \mu_n}(t) = \int_{\mathbb{R}^{n-1}} q_{n, \mu_n}(t, \lambda_2, \dots, \lambda_n) d\lambda_2 \dots d\lambda_n.$$

Let compute the two first moments of the measure ν_n :

$$m_1(\mu_n) = \int_{\mathbb{R}} t \nu_n(dt) = \frac{1}{n} \int_{\mathbb{R}^n} \sum_{k=1}^n \lambda_k \mathbb{P}_{n, \mu_n}(d\lambda) = 0,$$

the second moment is:

$$m_2(\nu_n) = \frac{1}{n} \int_{\mathbb{R}^n} \sum_{k=1}^n \lambda_k^2 \mathbb{P}_{n, \mu_n}(d\lambda),$$

Since for all $\alpha > 0$,

$$Z_n(\alpha) = \int_{\mathbb{R}^n} e^{-\alpha \sum_{k=1}^n \lambda_k^2} \prod_{k=1}^n |\lambda_k|^{2\mu_n} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta d\lambda_1 \cdots d\lambda_n = \alpha^{-n\mu_n - \frac{\beta}{4}n(n-1) - \frac{n}{2}} Z_n,$$

and

$$m_2(\nu_n) = -\frac{1}{n} \frac{d}{d\alpha} \log(Z_n(\alpha))|_{\alpha=1} = \mu_n + \frac{\beta}{4}(n-1) + \frac{1}{2}.$$

This suggests that ν_n does not converge, and that a scaling of order $\sqrt{n \frac{\beta}{4} + \mu_n}$ is necessary.

We come to The mean result: the measure ν_n converge weakly to some probability measure $\nu_{\beta,c}$ which is an equilibrium measure.

Theorem 4.1 *Let $(\mu_n)_n$ be a nonnegative real sequence, if $\lim_{n \rightarrow \infty} \frac{\mu_n}{n} = c$. Then the probability measure ν_n converge weakly to the probability $\nu_{\beta,c}$, where $\nu_{\beta,c}$ is the measure on $S = [-b, -a] \cup [a, b]$ with density with respect to the Lebesgue measure*

$$f_{\beta,c}(t) = \begin{cases} \frac{2}{\pi\beta} \frac{1}{|t|} \sqrt{(t^2 - a^2)(b^2 - t^2)} & \text{if } t \in S \\ 0 & \text{if } t \notin S \end{cases},$$

and $a = \sqrt{\frac{\beta}{2}} \sqrt{1 + \frac{2c}{\beta}} - \sqrt{1 + \frac{4c}{\beta}}$, $b = \sqrt{\frac{\beta}{2}} \sqrt{1 + \frac{2c}{\beta}} + \sqrt{1 + \frac{4c}{\beta}}$. Moreover the energy of the equilibrium measure $\nu_{\beta,c}$ is

$$E_{\beta,c}^* = \frac{3\beta}{8} + \frac{\beta}{4} \log\left(\frac{4}{\beta}\right) + c\left(\frac{3}{2} + \log\left(\frac{4}{\beta}\right)\right) + \frac{2c^2}{\beta} \log\left(\frac{4c}{\beta}\right) - \left(\frac{2c^2}{\beta} + c + \frac{\beta}{8}\right) \log\left(1 + \frac{4c}{\beta}\right).$$

The convergence is in the sense that for every continuous bounded function f on \mathbb{R}

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(t) \nu_n(dt) = \int_{\mathbb{R}} f(t) \nu_{\beta,c}(dt).$$

4.2 Equilibrium measure of generalized Gaussian unitary ensemble

For $c \geq 0$, $\beta > 0$, one considers on $\Sigma = \mathbb{R}$, the potential

$$Q_c(t) = t^2 + 2c \log \frac{1}{|t|},$$

The energy of a probability measure $\mu \in \mathfrak{M}^1(\mathbb{R})$ is defined by

$$E_{\beta,c}(\mu) = \frac{\beta}{2} \int_{\mathbb{R}^2} \log \frac{1}{|s-t|} \mu(ds) \mu(dt) + \int_{\mathbb{R}} Q_c(t) \mu(dt),$$

and let $U_{\beta,c}$ be the potential of the measure $\nu_{\beta,c}$, $U_{\beta,c}(x) = \int_{\mathbb{R}} \log \frac{1}{|x-y|} \nu_{\beta,c}(dy)$

Proposition 4.2 *The probability measure $\nu_{\beta,c}$ is the equilibrium measure, which mean that*

$$\inf \{E_{\beta,c}(\nu) \mid \nu \in \mathfrak{M}^1(\mathbb{R})\} = E(\nu_{\beta,c}) = E_{\beta,c}^*.$$

Furthermore

- (i) $U_{\beta,c}(x) + \frac{1}{2}Q_c(x) = C$, on S .
- (ii) $U_{\beta,c}(x) + \frac{1}{2}Q_c(x) \geq C$, on $\mathbb{R} \setminus S$.

We will give the value of the energy $E_{\beta,c}^*$ in section 3.3.

To prove the proposition we need same preliminary results and then applying proposition 2.2.

For more convenient notation we shall denote $c' = \frac{2c}{\beta}$.

Putting

$$f(z) = \frac{2}{\beta z} \sqrt{z-a} \sqrt{z-b} \sqrt{z+a} \sqrt{z+b},$$

The function f is holomorphic on the domain $\mathbb{C} \setminus (S \cup \{0\})$, of moderate growth near $S \cup \{0\}$.

Proposition 4.3 *The difference between the two limits values of f in the distribution sense, $[f] = f(x+i0) - f(x-i0)$, is given by*

$$[f] = 2i\pi\nu_{\beta,c'} + 2i\pi c' \delta_0.$$

Proof.

For $b > 0$, observe that the function $f(z) = \frac{2}{\beta z} \sqrt{z-a} \sqrt{z-b} \sqrt{z+a} \sqrt{z+b}$, is defined and holomorphic on $\mathbb{C} \setminus]-\infty, b]$.

For $x > b$, $f(x) = \frac{2}{\beta x} \sqrt{(x-a)(x-b)(x+a)(x+b)} = \frac{2}{\beta x} \sqrt{(x^2-a^2)(x^2-b^2)}$, be

the usual square root of positive numbers.

For $x < -b$,

$$\lim_{\varepsilon \rightarrow 0, \varepsilon > 0} f(x \pm i\varepsilon) = e^{\pm i\pi} e^{\pm i\pi} \frac{2}{\beta x} \sqrt{(a-x)(b-x)(-x-a)(-x-b)} = \frac{2}{\beta x} \sqrt{(x^2 - a^2)(x^2 - b^2)}.$$

There for f extended as holomorphic function on $\mathbb{C} \setminus S$. Furthermore For $-b < x < -a$,

$$\lim_{\varepsilon \rightarrow 0, \varepsilon > 0} f(x \pm i\varepsilon) = -e^{\pm i\frac{\pi}{2}} \frac{2}{\beta x} \sqrt{(a-x)(b-x)(-x-a)(x+b)} = \mp i \frac{2}{\beta x} \sqrt{(a^2 - x^2)(b^2 - x^2)}.$$

For $-a < x < a$, $x \neq 0$,

$$\lim_{\varepsilon \rightarrow 0, \varepsilon > 0} f(x \pm i\varepsilon) = e^{\pm i\pi} \frac{2}{\beta x} \sqrt{(a-x)(b-x)(x+a)(x+b)} = -\frac{2}{\beta x} \sqrt{(a^2 - x^2)(b^2 - x^2)}.$$

For z near 0, by Taylor expansion

$$f(z) = -\frac{2ab}{\beta z} + g(z) = \frac{-c'}{z} + g(z),$$

where g is an holomorphic function.

For $a < x < b$,

$$\lim_{\varepsilon \rightarrow 0, \varepsilon > 0} f(x \pm i\varepsilon) = e^{\pm i\frac{\pi}{2}} \frac{2}{\beta x} \sqrt{(a-x)(b-x)(-x-a)(x+b)} = \pm i \frac{2}{\beta x} \sqrt{(x^2 - a^2)(b^2 - x^2)}.$$

It follows that

$$[f] = 2i \frac{2}{\beta|x|} \sqrt{(x^2 - a^2)(b^2 - x^2)} \chi_S + 2i\pi c' Y_0 = 2i\pi \nu_{\beta, c'} + 2i\pi c' \delta_0.$$

Which complete the proof of the proposition. ■

Let denote by $G_{\beta, c'}$ the Cauchy transform of the measure $\nu_{\beta, c'}$: for all $z \in \mathbb{C} \setminus S$,

$$G_{\beta, c'}(z) = \int_{\mathbb{R}} \frac{1}{z - t} \nu_{\beta, c'}(dt).$$

Proposition 4.4 *The Cauchy transform of the measure $\nu_{\beta, c'}$ is defined on $\mathbb{C} \setminus S$,*

$$G_{\beta, c'}(z) = -f(z) + \frac{2}{\beta} z - \frac{c'}{z}$$

Proof. From the previous proposition we have for all $x \in S$

$$\lim_{\varepsilon \rightarrow 0, \varepsilon > 0} (f(x + i\varepsilon) - f(x - i\varepsilon)) = 2i \frac{2}{\beta x} \sqrt{(x^2 - a^2)(b^2 - x^2)}.$$

It follows that, if φ is a holomorphic function in a neighborhood U of S , and γ is a path in U around S in the positive sense, then

$$\frac{1}{2i\pi} \int_{\gamma} \varphi(\omega) f(\omega) d\omega = \int_S \varphi(x) \nu_{\beta, c'}(dx).$$

in particular, for

$$\varphi(\omega) = \frac{1}{z - \omega},$$

if z is in the exterior of γ , then

$$\frac{1}{2i\pi} \int_{\gamma} \frac{1}{z - \omega} f(\omega) d\omega = G_{\beta, c'}(z).$$

We will use the theorem of residues to derive the expression of $G_{c'}$.

The function $g(\omega) = \frac{1}{z - \omega} f(\omega) = \frac{2}{\beta \omega(z - \omega)} \sqrt{\omega - a} \sqrt{\omega - b} \sqrt{\omega + a} \sqrt{\omega + b}$, is meromorphic in $\overline{\mathbb{C}} \setminus S \cup \{0, z\}$. with simple pole at $\omega = 0$, $\omega = z$ and a pole at infinity.

Furthermore the residue at $\omega = 0$ is $-\frac{2c}{\beta z} = -\frac{c'}{z}$, the residue at $\omega = z$ is $-f(z)$, and f admit a Laurent expansion for $|\omega| > \max(|a|, |b|, |z|)$

$$f(\omega) = \frac{2\omega}{\beta} \sqrt{1 - \frac{a}{\omega}} \sqrt{1 - \frac{b}{\omega}} \sqrt{1 + \frac{a}{\omega}} \sqrt{1 + \frac{b}{\omega}} = \frac{2}{\beta} \omega - \frac{a^2 + b^2}{\beta} \frac{1}{\omega} + \dots$$

then the residue at $\omega = \infty$ is $-\frac{2z}{\beta}$.

Which give

$$G_{\beta, c'}(z) = -f(z) + \frac{2}{\beta} z - \frac{c'}{z}.$$

Proof of proposition 3.2.

Let denote by $U_{\beta, c'}$ the logarithmic potential of the measure $\nu_{\beta, c'}$: for all $x \in \mathbb{R}$,

$$U_{\beta, c'}(x) = \int_{\mathbb{R}} \log \frac{1}{|x - t|} \nu_{\beta, c'}(dt).$$

The function $U_{\beta,c'}$ is even and

$$\frac{d}{dx}U_{\beta,c'}(x) = -\text{Re}G_{\beta,c'}(x).$$

We will study the variation of the function

$$\varphi(x) = U_{\beta,c'}(x) + \frac{1}{2}Q_{c'}(x).$$

The function φ is even and

$$\varphi'(x) = -\text{Re}G_{\beta,c'}(x) + \frac{1}{2}\frac{d}{dx}Q_{c'}(x).$$

(It is not defined on the point $x = 0$). The last function vanished on S , therefore the function is constant on each connect components of S . Since the function φ is even therefore the constant is the same on each components. Let denoted it by C .

x	$-b$	$-a$	0	a	b
$\varphi'(x)$	$-$	0	$+$	$-$	0
$\varphi(x)$	\searrow	\nearrow		\searrow	\nearrow
	C			C	

Therefore

$$\begin{aligned} U_{\beta,c'}(x) + \frac{1}{2}Q_{c'}(x) &\geq C \quad \text{in } \mathbb{R}, \\ &= C \quad \text{in } S. \end{aligned}$$

By making use the proposition 2.2 the equilibrium measure ν^* coincide with $\nu_{\beta,c'}$. ■

4.3 Energy of equilibrium measure

Consider the integral,

$$A_n = \int_{\mathbb{R}^n} e^{-n \sum_{k=1}^n \lambda_k^2} \prod_{k=1}^n |\lambda_k|^{2\mu_n} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta d\lambda_1 \cdots d\lambda_n = \int_{\mathbb{R}^n} e^{-K_n(\lambda) - \sum_{i=1}^n Q_{a_n}(\lambda_i)} d\lambda_1 \cdots d\lambda_n. \quad (4.2)$$

where

$$K_n(\lambda) = K_n(\lambda_1, \dots, \lambda_n) = \frac{\beta}{2} \sum_{i \neq j} \log \frac{1}{|\lambda_i - \lambda_j|} + (n-1) \sum_{i=1}^n Q_{\alpha_n}(\lambda_i),$$

and

$$Q_{\alpha_n}(x) = x^2 + 2\alpha_n \log \frac{1}{|x|}, \quad \alpha_n = \frac{\mu_n}{n}$$

For $c \geq 0$ consider also the integral

$$B_n = \int_{\mathbb{R}^n} e^{-n \sum_{k=1}^n \lambda_k^2} \prod_{k=1}^n |\lambda_k|^{2nc} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta d\lambda_1 \cdots d\lambda_n = \int_{\mathbb{R}^n} e^{-n^2 V_{n,c}(\lambda)} d\lambda_1 \cdots d\lambda_n. \quad (4.3)$$

where

$$Q_c(x) = x^2 + 2c \log \frac{1}{|x|},$$

Recall that the energy for a probability ν is defined by

$$E_{\beta,\delta}(\nu) = \frac{\beta}{2} \int_{\mathbb{R}^2} \log \frac{1}{|x-y|} \nu(dx) \nu(dy) + \int_{\mathbb{R}} Q_\delta(x) \nu(dx),$$

where

$$Q_\delta(x) = x^2 + 2\delta \log \frac{1}{|x|},$$

We saw that

$$\lim_{n \rightarrow \infty} -\frac{1}{n^2} \log B_n = E_{\beta,c}^*.$$

See for instance (Faraut [4]). We will prove this result in proposition 4.6. for more general potential

Remark that $\lim_{x \rightarrow \pm\infty} K_n(x) = +\infty$ and $\lim_{x \rightarrow 0} K_n(x) = +\infty$, the same hold in the diagonal of \mathbb{R}^n . Since the function K_n is continuous except on the diagonal and 0 where it has as limit $+\infty$. Hence it is bounded below and the minimum is realized at some point $\lambda^{(n)} = (\lambda_1^{(n)}, \dots, \lambda_n^{(n)})$, which means that

$$\inf_{\mathbb{R}^n} K_n(x) = K_n(\lambda^{(n)}).$$

Let denote by

$$\tau_n = \frac{1}{n(n-1)} \inf_{x \in \mathbb{R}^n} K_n(x) \quad \text{and} \quad \rho_n = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i^{(n)}},$$

$\delta_{\lambda_i^{(n)}}$ is the Dirac mass at $\lambda_i^{(n)}$.

From proposition 4.2, if we replace c by α_n the equilibrium measure of the potential $\frac{2}{\beta}Q_{\alpha_n}$ is ν_{β,α_n} , where the density of the equilibrium measure ν_{β,α_n} is given by

$$f_{\beta,\alpha_n}(t) = \begin{cases} \frac{2}{\pi\beta} \frac{1}{|t|} \sqrt{(t^2 - a_n^2)(b_n^2 - t^2)} & \text{if } t \in S_n \\ 0 & \text{if } t \notin S_n \end{cases},$$

$$S_n = [-b_n, a_n] \cup [a_n, b_n] \text{ and } a_n = \sqrt{\frac{\beta}{2}} \sqrt{1 + \frac{\alpha_n}{\beta}} - \sqrt{1 + \frac{2\alpha_n}{\beta}}, b_n = \sqrt{\frac{\beta}{2}} \sqrt{1 + \frac{\alpha_n}{\beta}} + \sqrt{1 + \frac{2\alpha_n}{\beta}}.$$

Lemma 4.5 *Let $(\mu_n)_n$ be a positive real sequence. Assume there is some constant c such that $\lim_{n \rightarrow \infty} \frac{\mu_n}{n} = c$. Then*

(1) *The probability measure ν_{β,α_n} converge weakly to the probability $\nu_{\beta,c}$.*

(2) $E_{\beta,c}^* = \lim_{n \rightarrow \infty} E_{\beta,\alpha_n}$.

where $E_{\beta,c}^*$ is the energy of the equilibrium measure $\nu_{\beta,c}$.

Proposition 4.6 *Let $(\mu_n)_n$ be a positive real sequence. Assume there is some constant c such that $\lim_{n \rightarrow \infty} \frac{\mu_n}{n} = c$. Then*

(1) $\lim_{n \rightarrow \infty} \tau_n = E_{\beta,c}^*$.

(2) *The measure ρ_n converge weakly to the equilibrium measure $\nu_{\beta,c}$.*

(3) $\lim_{n \rightarrow \infty} -\frac{1}{n^2} \log A_n = E_{\beta,c}^*$.

Proposition 4.7 *Let $(\mu_n)_n$ be a positive real sequence. Assume there is some constant c such that $\lim_{n \rightarrow \infty} \frac{\mu_n}{n} = c$. Then the energy $E_{\beta,c}^*$ is given by*

$$E_{\beta,c}^* = \frac{3\beta}{8} + \frac{\beta}{4} \log\left(\frac{4}{\beta}\right) + c\left(\frac{3}{2} + \log\frac{4}{\beta}\right) + \frac{2c^2}{\beta} \log\frac{4c}{\beta} - \left(\frac{2c^2}{\beta} + c + \frac{\beta}{8}\right) \log\left(1 + \frac{4c}{\beta}\right).$$

For $c = 0$, one recover's the energy of the β -Gaussian unitary ensemble

$$E_{\beta,0}^* = \frac{3\beta}{8} + \frac{\beta}{4} \log\left(\frac{4}{\beta}\right).$$

Proof of lemma 3.5.

Step(1): The probability measures ν_{β,α_n} and $\nu_{\beta,c}$ have density respectively f_{β,α_n} and f_c . It is easy to see that the density f_{β,α_n} converges Pointwise to the density f_c .

Then by applying Fatou lemma we deduce the convergence in the weak topology.

Step(2) : We know by definition of the energy that

$$E_{\beta,c}^* = \inf_{\nu \in \mathfrak{M}^1(\mathbb{R})} E_{\beta,c}(\nu) \leq E_{\beta,c}(\nu_{\beta,\alpha_n}), \quad (4.4)$$

and

$$E_{\beta,c}(\nu_{\beta,\alpha_n}) = E_{\beta,\alpha_n}(\nu_{\beta,\alpha_n}) + \int_{\mathbb{R}} (Q_c(x) - Q_{\alpha_n}) \nu_{\beta,\alpha_n}(dx),$$

which can be writing as

$$E_{\beta,c}(\nu_{\beta,\alpha_n}) = E_{\beta,\alpha_n}^* + \int_{\mathbb{R}} (Q_c(x) - Q_{\alpha_n}) \nu_{\beta,\alpha_n}(dx), \quad (4.5)$$

where $E_{\beta,\alpha_n}^* = \inf_{\nu \in \mathfrak{M}^1(\mathbb{R})} E_{\beta,\alpha_n}(\nu) = E_{\beta,\alpha_n}(\nu_{\beta,\alpha_n})$.

Furthermore

$$E_{\beta,\alpha_n}^* = \inf_{\nu \in \mathfrak{M}^1(\mathbb{R})} E_{\beta,\alpha_n}(\nu) \leq E_{\beta,\alpha_n}(\nu_{\beta,c}), \quad (4.6)$$

and

$$E_{\beta,\alpha_n}(\nu_{\beta,c}) = E_{\beta,c}(\nu_{\beta,c}) + \int_{\mathbb{R}} (Q_{\alpha_n}(x) - Q_c(x)) \nu_{\beta,c}(dx),$$

$$E_{\beta,\alpha_n}(\nu_{\beta,c}) = E_{\beta,c}^* + \int_{\mathbb{R}} (Q_{\alpha_n}(x) - Q_c(x)) \nu_{\beta,c}(dx). \quad (4.7)$$

From equations (3.3), (3.4), (3.5), (3.6) one gets

$$E_{\beta,c}^* + \int_{\mathbb{R}} (Q_{\alpha_n}(x) - Q_c(x)) \nu_{\beta,\alpha_n}(dx) \leq E_{\beta,\alpha_n}^* \leq E_{\beta,c}^* + \int_{\mathbb{R}} (Q_{\alpha_n}(x) - Q_c(x)) \nu_{\beta,c}(dx) \quad (4.8)$$

So it is enough to prove that the integrals go to 0 when n go to infinity. Recall that the probability measures ν_{β,α_n} and $\nu_{\beta,c}$ are supported respectively by S_n and S .

Furthermore

$$|Q_{\alpha_n}(x) - Q_c(x)| = 2|\alpha_n - c| |\log |x||,$$

Since the sequence b_n converge to b hence there is some positive constant C such that for all $n \in \mathbb{N}$,

$$\sup_{S_n \cup S} |\log |x|| = \max(\log b_n, \log b) \leq C.$$

Take the limit in equation (3.7) and use the facts that ν_{β,α_n} and $\nu_{\beta,c}$ are probability measures and the sequence α_n converge to c we deduce that

$$\lim_{n \rightarrow \infty} E_{\beta,\alpha_n}^* = E_{\beta,c}^*.$$

Proof of proposition 4.6.

We will denote

$$k_\delta(s, t) = \log \frac{1}{|s - t|} + \frac{1}{2} Q_\delta(s) + \frac{1}{2} Q_\delta(t),$$

for $\ell > 0$,

$$k_\delta^\ell(s, t) = \inf(k_\delta(s, t), \ell).$$

and

$$h_{\alpha_n}(t) = Q_{\alpha_n}(t) - \log(1 + t^2).$$

Step(1): In this step we will prove (1) and (2).

Let $\gamma \in \mathfrak{M}^1(\mathbb{R})$ be a probability measure then

$$\begin{aligned} \int_{\mathbb{R}^n} K_n(x) \gamma(dx_1) \dots \gamma(dx_n) &= \frac{\beta}{2} n(n-1) \int_{\mathbb{R}^2} \log \frac{1}{|s - t|} \gamma(ds) \gamma(dt) + n(n-1) \int_{\mathbb{R}} Q_{\alpha_n}(s) \gamma(ds) \\ &= n(n-1) E_{\beta, \alpha_n}(\gamma). \end{aligned}$$

Then

$$\tau_n \leq E_{\beta, \alpha_n}(\gamma),$$

for $\gamma = \nu_{\beta, \alpha_n}$ which is the equilibrium measure for the potential Q_{α_n} , we obtains

$$\tau_n \leq E_{\beta, \alpha_n}^*,$$

By using step 2 of lemma 3.5 we deduce

$$\limsup_n \tau_n \leq \lim E_{\beta, \alpha_n}^* = E_{\beta, c}^*. \quad (4.9)$$

Furthermore

$$\begin{aligned} E_{\beta, \alpha_n}^\ell(\rho_n) &= \int_{\mathbb{R}^2} k_{\alpha_n}^\ell(s, t) \rho_n(ds) \rho_n(dt) \\ &= \frac{1}{n^2} \sum_{i, j=1}^n k_{\beta, \alpha_n}^\ell(\lambda_i^{(n)}, \lambda_j^{(n)}) \\ &\leq \frac{1}{n^2} \sum_{1 \leq i \neq j \leq n} k_{\alpha_n}(\lambda_i^{(n)}, \lambda_j^{(n)}) + \frac{\ell}{n} \\ &= \frac{1}{n^2} K_n(\lambda^{(n)}) + \frac{\ell}{n} \\ &= \frac{n(n-1)}{n^2} \tau_n + \frac{\ell}{n} \leq E_{\beta, \alpha_n}^* + \frac{\ell}{n}. \end{aligned}$$

By the inequality

$$|s - t| \leq \sqrt{1 + s^2} \sqrt{1 + t^2},$$

it follows that

$$\frac{1}{2}h_{\alpha_n}(s) + \frac{1}{2}h_{\alpha_n}(t) \leq k_{\alpha_n}(s, t), \quad (4.10)$$

and then

$$\int_{\mathbb{R}} h_{\alpha_n}(s) \rho_n(ds) \leq E_{\beta, \alpha_n}^{\ell}(\rho_n) \leq E_{\beta, \alpha_n}^* + \frac{\ell}{n}$$

Since the sequence $E_{\beta, \alpha_n}^* + \frac{\ell}{n}$ is bounded uniformly on n by some positive constant C_0 . Furthermore

$$h_{\alpha_n}(s) = Q_{\alpha_n}(s) - \log(1 + s^2) = s^2 + \alpha_n \log \frac{1}{|s|} - \log(1 + s^2).$$

Since the positive sequence α_n converge to c , then there is two positive constants a_1, a_2 such that $a_1 \leq \alpha_n \leq a_2$ and

$$h_{\alpha_n}(s) \geq s^2 + a_1 \log \frac{1}{|s|} - \log(1 + s^2) = h_1(s) \quad \text{if } 0 < |s| \leq 1$$

$$h_{\alpha_n}(s) \geq s^2 + a_2 \log \frac{1}{|s|} - \log(1 + s^2) = h_2(s) \quad \text{if } |s| \geq 1$$

Let $h(s) = \inf(h_1(s), h_2(s))$, then $\lim_{|s| \rightarrow \infty} h(s) = +\infty$ and

$$\int_{\mathbb{R}} h(s) \rho_n(ds) \leq C_0.$$

Hence by Prokhorov criterium this proves that the sequence $(\rho_n)_n$ is relatively compact for the weak topology. Therefore there is a converging subsequence: ρ_{n_k} to ρ which means, for all bound continuous functions on \mathbb{R}

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x) \rho_{n_k}(dx) = \int_{\mathbb{R}} f(x) \rho(dx).$$

We will denote by ρ_n the subsequence.

For $\ell > 0$ consider as in the previous the kernel $k_{\alpha_n}^{\ell}(s, t) = \inf(k_{\alpha_n}(s, t), \ell)$ and $k_c^{\ell}(s, t) = \inf(k_c(s, t), \ell)$.

Let $\varepsilon > 0$, there is n_0 , such that for all $n \geq n_0$,

$$c - \varepsilon \leq \alpha_n \leq c + \varepsilon,$$

Let $n \geq n_0$, divided $\Sigma = \mathbb{R}^2 \setminus \{(s, t) \mid s = t \text{ or } s = 0 \text{ or } t = 0\}$ to fourth region

$$R_1 = \{(s, t) \in \Sigma \mid |s| \geq 1 \text{ and } |t| \geq 1\}, \quad R_2 = \{(s, t) \in \Sigma \mid 0 < |s| \leq 1 \text{ and } 0 < |t| \leq 1\},$$

and

$$R_3 = \{(s, t) \in \Sigma \mid 0 < |s| \leq 1 \text{ and } |t| \geq 1\}, \quad R_4 = \{(s, t) \in \Sigma \mid |s| \geq 1 \text{ and } 0 < |t| \leq 1\}.$$

If $(s, t) \in R_1$, then

$$k_{\alpha_n}(s, t) \geq k_{c+\varepsilon}(s, t).$$

If $(s, t) \in R_2$,

$$k_{\alpha_n}(s, t) \geq k_{c-\varepsilon}(s, t).$$

If $(s, t) \in R_3$,

$$k_{\alpha_n}(s, t) \geq \log \frac{1}{|s-t|} + \frac{1}{2} Q_{c+\varepsilon}(t) + \frac{1}{2} Q_{c-\varepsilon}(s),$$

hence

$$k_{\alpha_n}(s, t) \geq \frac{1}{2} (k_{c+\varepsilon}(s, t) + k_{c-\varepsilon}(s, t)).$$

By symmetry of the kernel k_{α_n} the last inequality is valid in R_4 .

we obtain for $(s, t) \in \Sigma$,

$$k_{\alpha_n}(s, t) \geq a k_{c+\varepsilon}(s, t) + b k_{c-\varepsilon}(s, t),$$

where $(a, b) = (1, 0)$ in R_1 , $(a, b) = (0, 1)$ in R_2 and $(a, b) = (\frac{1}{2}, \frac{1}{2})$ in $R_3 \cup R_4$. Hence if we take the infimum we obtain

$$k_{\alpha_n}^\ell(s, t) \geq a k_{c+\varepsilon}^\ell(s, t) + b k_{c-\varepsilon}^\ell(s, t).$$

Moreover for the energy one gets, for all $n \geq n_0$

$$a E_{\beta, c+\varepsilon}^\ell(\rho_n) + b E_{\beta, c-\varepsilon}^\ell(\rho_n) \leq E_{\beta, \alpha_n}^\ell(\rho_n).$$

Which gives

$$a E_{\beta, c+\varepsilon}^\ell(\rho_n) + b E_{\beta, c-\varepsilon}^\ell(\rho_n) \leq \frac{n(n-1)}{n^2} \tau_n + \frac{\ell}{n}.$$

As n goes to infinity we obtain

$$\liminf_n \left(a E_{\beta, c+\varepsilon}^\ell(\rho_n) + b E_{\beta, c-\varepsilon}^\ell(\rho_n) \right) \leq \liminf \tau_n,$$

hence by the weak convergence of the subsequence ρ_n it follow

$$a E_{\beta, c+\varepsilon}^\ell(\rho) + b E_{\beta, c-\varepsilon}^\ell(\rho) \leq \liminf \tau_n,$$

applying the monotone convergence theorem, when ℓ goes to 0, it follows that

$$a E_{\beta, c+\varepsilon}(\rho) + b E_{\beta, c-\varepsilon}(\rho) \leq \liminf \tau_n.$$

Since ρ is a probability measure and using the values of a, b we obtain $a E_{\beta, c+\varepsilon}(\rho) + b E_{\beta, c-\varepsilon}(\rho) = E_{\beta, c}(\rho)$. hence

$$E_{\beta, c}(\rho) \leq \liminf \tau_n.$$

Furthermore

$$\inf_{\mu \in \mathfrak{M}^1(\mathbb{R})} E_{\beta,c}(\mu) \leq E_{\beta,c}(\rho).$$

We saw from proposition 4.2. that the minimum is realized at the probability measure $\nu_{\beta,c}$ and the minimum is $E_{\beta,c}^*$. Hence

$$E_{\beta,c}^* \leq E_{\beta,c}(\rho) \leq \liminf \tau_n.$$

It follows that

$$E_{\beta,c}^* \leq E_{\beta,c}(\rho) \leq \liminf \tau_n \leq \limsup \tau_n \leq E_{\beta,c}^*,$$

in the last inequalities we used equation (4.13). Therefore

$$E(\rho) = E_{\beta,c}^* = E_{\beta,c}(\nu_{\beta,c}).$$

This implies that $\rho = \nu_{\beta,c}$. We have proved that $\nu_{\beta,c}$ is the only possible limit for a subsequence of the sequence (ρ_n) . It follows that the sequence (ρ_n) itself converges: for all bounded continuous function

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x) \rho_n(dx) = \int_{\mathbb{R}} f(x) \nu_{\beta,c}(dx),$$

and

$$\lim_{n \rightarrow \infty} \tau_n = E_{\beta,c}^*.$$

Step (2): Now we will prove: $\lim_{n \rightarrow \infty} -\frac{1}{n^2} \log A_n = E_{\beta,c}^*$.

Recall that

$$A_n = \int_{\mathbb{R}^n} e^{-K_n(\lambda) - \sum_{i=1}^n Q_{\alpha_n}(\lambda_i)} d\lambda_1 \cdots d\lambda_n,$$

it follows that

$$A_n \leq e^{-n(n-1)\tau_n} \left(\int_{\mathbb{R}} e^{-Q_{\alpha_n}(\lambda)} d\lambda \right)^n = e^{-n(n-1)\tau_n} \left(\Gamma(\alpha_n + \frac{1}{2}) \right)^n,$$

and

$$\frac{1}{n^2} \log A_n \leq -\frac{n-1}{n} \tau_n + \frac{1}{n} \log \Gamma(\alpha_n + \frac{1}{2}).$$

Since the sequence (α_n) converge to c then $\lim_{n \rightarrow \infty} \log \Gamma(\alpha_n + \frac{1}{2}) = \log \Gamma(c + \frac{1}{2})$ which gives

$$\liminf_n -\frac{1}{n^2} \log A_n \geq \liminf_n \tau_n = E_{\beta,c}^*. \quad (4.11)$$

Furthermore if μ is a probability measure then

$$\int_{\mathbb{R}^n} K_n(x) \mu(dx_1) \cdots \mu(dx_n) = n(n-1)E_{\beta, \alpha_n}(\mu),$$

Let $\mu(dt) = \nu_{\beta, c}(dt) = f_{\beta, c}(t)dt$ supported by $S = [-b, -a] \cup [a, b]$, the function $f_{\beta, c}(t) > 0$ except on subset of S with measure zero. Applying Jensen inequality to the exponential function then

$$\begin{aligned} A_n &= \int_{\mathbb{R}^n} \exp \left(-K_n(x) - \sum_{i=1}^n Q_{\alpha_n}(x_i) - \sum_{i=1}^n \log f_c(x_i) \right) \prod_{i=1}^n f_c(x_i) dx_1 \cdots dx_n \\ &\geq \exp \left(\int_{\mathbb{R}^n} \left(-K_n(x) - \sum_{i=1}^n Q_{\alpha_n}(x_i) - \sum_{i=1}^n \log f_c(x_i) \right) \prod_{i=1}^n f_c(x_i) dx_1 \cdots dx_n \right) \\ &\geq e^{-n(n-1)E_{\beta, \alpha_n}(\nu_{\beta, c})} \exp \left(-n \int_{\mathbb{R}} Q_{\alpha_n}(x) f_{\beta, c}(x) dx \right) \exp \left(-n \int_{\mathbb{R}} f_{\beta, c}(x) \log f_{\beta, c}(x) dx \right). \end{aligned}$$

From lemme 4.5 we have

$$\lim_{n \rightarrow \infty} E_{\beta, \alpha_n}(\nu_{\beta, c}) = E_{\beta, c}(\nu_{\beta, c}) = E_{\beta, c}^*,$$

and

$$\int_{\mathbb{R}} Q_{\alpha_n}(x) f_c(x) dx = 2 \int_a^b Q_{\alpha_n}(x) f_c(x) dx \leq 2(b^2 + 2|\alpha_n| \log b),$$

furthermore the last integral exist by the continuity of the function $x \log x$ near 0 and the continuous function f_c is with compactly support S . So the integral is bounded by some constant say M . Then

$$-\frac{1}{n^2} \log A_n \leq \frac{n-1}{n} E_{\beta, c}^* + \frac{1}{n} (2b^2 + 4|\alpha_n| \log b + M).$$

It follows that

$$\limsup_n -\frac{1}{n^2} \log A_n \leq \limsup_n \left(\frac{n-1}{n} E_{\beta, \alpha_n}(\nu_{\beta, c}) + \frac{1}{n} (b^2 + |\alpha_n| \log a + M) \right).$$

Since α_n converge. Hence

$$\limsup_n -\frac{1}{n^2} \log A_n \leq E_{\beta, c}^*. \quad (4.12)$$

Equations (4.11) and (4.12) gives that

$$\lim_n -\frac{1}{n^2} \log A_n = E_{\beta, c}^*.$$

Which complete the proof. ■

If we choose $\mu_n = nc$ we obtains the same result for the sequence B_n ,

$$\lim_n -\frac{1}{n^2} \log B_n = E_{\beta, c}^*.$$

Proof of proposition 4.7. For more convenient we will prove the proposition first when $\beta = 2$ and then deduce from proposition 3.3. the result for all $\beta > 0$

First case $\beta = 2$. By performing the change of variable $x_k = \lambda_k \sqrt{n}$ in the expression of A_n equation (3.2), we obtain

$$A_n = n^{-n\mu_n - \frac{n^2}{2}} n! C_n = n^{-n\mu_n - \frac{n^2}{2}} n! \prod_{k=1}^{n-1} \gamma_{\mu_n}(k),$$

where $\gamma_{\mu_n}(k)$ is defined in equation (3.1).

First step. Let $n = 2m$ be an even integer. Then

$$\begin{aligned} A_{2m} &= (2m)! (2m)^{-2m\mu_{2m} - \frac{(2m)^2}{2}} \prod_{k=1}^{2m-1} \gamma_{\mu_{2m}}(k) \\ &= (2m)! (2m)^{-2m\mu_{2m} - \frac{(2m)^2}{2}} \prod_{k=0}^{m-1} \gamma_{\mu_{2m}}(2k) \prod_{k=1}^{m-1} \gamma_{\mu_{2m}}(2k+1) \\ &= (2m)! (2m)^{-2m\mu_{2m} - \frac{(2m)^2}{2}} \prod_{k=0}^{m-1} (k!)^2 \left(\Gamma(k + \mu_{2m} + \frac{1}{2}) \right)^2 \prod_{k=1}^{m-1} (k + \mu_{2m} + \frac{1}{2}), \end{aligned}$$

in the last equality we use the fact that $\Gamma(x+1) = x\Gamma(x)$.

Take the logarithm of A_{2m}

$$\begin{aligned} \log(A_{2m}) &= \sum_{k=1}^{2m} \log(k) + 2 \sum_{k=1}^{m-1} (m-k) \log(k) + 2 \sum_{k=0}^{m-1} \log \Gamma(k + \mu_{2m} + \frac{1}{2}) \\ &\quad + \sum_{k=0}^{m-1} \log(k + \mu_{2m} + \frac{1}{2}) - (2m\mu_{2m} + \frac{(2m)^2}{2}) \log(2m). \end{aligned}$$

It is easy to see that for m large enough

$$\sum_{k=1}^{2m} \log(k) = o(m^2). \tag{4.13}$$

Furthermore from the Stiriling asymptotic formula we have, for $0 \leq k \leq m-1$

$$\log \Gamma(k + \mu_{2m} + \frac{1}{2}) = (k + \mu_{2m}) \log(k + \mu_{2m} + \frac{1}{2}) - (k + \mu_{2m} + \frac{1}{2}) + o\left(\log(k + \mu_{2m} + \frac{1}{2})\right), \tag{4.14}$$

and by the fact that $\mu_n = cn + o(n)$, we deduce, that

$$\log(k + \mu_{2m} + \frac{1}{2}) = \log(k + \mu_{2m}) + \log(1 + \frac{1}{2(k + \mu_{2m})}) = \log(k + \mu_{2m}) + \frac{1}{2(k + \mu_{2m})} + o(\frac{1}{m}),$$

$$\text{and } \log(k + \mu_{2m} + \frac{1}{2}) = o(m), \quad \sum_{k=0}^{m-1} (k + \mu_{2m} + \frac{1}{2}) = \sum_{k=0}^{m-1} (k + \mu_{2m}) + o(m^2).$$

By summing both side of (4.14), one gets

$$\sum_{k=0}^{m-1} \log \Gamma(k + \mu_{2m} + \frac{1}{2}) = \sum_{k=0}^{m-1} (k + \mu_{2m}) \log(k + \mu_{2m}) - \sum_{k=0}^{m-1} (k + \mu_{2m}) + o(m^2), \quad (4.15)$$

and

$$\sum_{k=0}^{m-1} \log(k + \mu_{2m} + \frac{1}{2}) = o(m^2). \quad (4.16)$$

Hence, from equation (4.13), (4.15) and (4.16), it follows

$$\begin{aligned} \frac{1}{(2m)^2} \log(A_{2m}) &= -(\frac{\mu_{2m}}{2m} + \frac{1}{2}) \log(2m) - \frac{2}{(2m)^2} \left(\frac{m(m-1)}{2} + m\mu_{2m} \right) \\ &\quad + \frac{1}{2m} \sum_{k=1}^{m-1} (1 - \frac{k}{m}) \log(k) + \frac{1}{2m^2} \sum_{k=0}^{m-1} (k + \mu_{2m}) \log(k + \mu_{2m}) + o(1). \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{(2m)^2} \log(A_{2m}) &= -(\frac{\mu_{2m}}{2m} + \frac{1}{2}) \log(2m) - (\frac{1}{2} + \frac{\mu_{2m}}{2m}) + \frac{1}{2m} \sum_{k=1}^{m-1} (1 - \frac{k}{m}) \log(m) \\ &\quad + \frac{1}{2m^2} \sum_{k=0}^{m-1} (k + \mu_{2m}) \log(m + \mu_{2m}) + S_m^1 + S_m^2 + o(1), \end{aligned}$$

where

$$S_m^1 = \frac{1}{2m} \sum_{k=1}^{m-1} (1 - \frac{k}{m}) \log(\frac{k}{m}),$$

and

$$S_m^2 = \frac{1}{2m^2} \sum_{k=0}^{m-1} (k + \mu_{2m}) \log(\frac{k + \mu_{2m}}{m + \mu_{2m}}).$$

Applying Riemann sums for both sums S_m^1 and S_m^2 , we obtain

$$\lim_{m \rightarrow \infty} S_m^1 = \lim_{m \rightarrow \infty} \frac{1}{2m} \sum_{k=1}^{m-1} (1 - \frac{k}{m}) \log(\frac{k}{m}) = \frac{1}{2} \int_0^1 (1-x) \log x dx = -\frac{3}{8}, \quad (4.17)$$

$$\begin{aligned}
\lim_{m \rightarrow \infty} S_m^2 &= \lim_{m \rightarrow \infty} \frac{1}{2} \left(1 + \frac{\mu_{2m}}{m}\right)^2 \frac{1}{m + \mu_{2m}} \sum_{k=0}^{m-1} \left(\frac{k + \mu_{2m}}{m + \mu_{2m}}\right) \log\left(\frac{k + \mu_{2m}}{m + \mu_{2m}}\right) \\
&= \frac{1}{2} (1 + 2c)^2 \int_{\frac{2c}{1+2c}}^1 x \log x dx = -\frac{1}{8} (1 + 2c)^2 + \frac{1}{2} c^2 + c^2 \log\left(1 + \frac{1}{2c}\right).
\end{aligned} \tag{4.18}$$

Now we will compute the limits of the others terms

$$I_m = -\left(\frac{\mu_{2m}}{2m} + \frac{1}{2}\right) \log(2m) + \frac{1}{2m} \sum_{k=1}^{m-1} \left(1 - \frac{k}{m}\right) \log(m) + \frac{1}{2m^2} \sum_{k=0}^{m-1} (k + \mu_{2m}) \log(m + \mu_{2m}) - \left(\frac{1}{4} + \frac{\mu_{2m}}{2m}\right)$$

By simple computation it yields

$$\begin{aligned}
I_m &= -\left(\frac{\mu_{2m}}{2m} + \frac{1}{2}\right) \log(2m) + \frac{m-1}{4m} \log(m) + \frac{1}{2m^2} \left(\frac{m(m-1)}{2} + m\mu_{2m}\right) \log(m + \mu_{2m}) - \left(\frac{1}{4} + \frac{\mu_{2m}}{2m}\right) \\
I_m &= -\left(\frac{\mu_{2m}}{2m} + \frac{1}{2}\right) \log 2 + \left(\frac{m-1}{2m} - \frac{1}{2}\right) \log(m) + \left(\frac{m-1}{4m} + \frac{\mu_{2m}}{2m}\right) \log\left(1 + \frac{\mu_{2m}}{m}\right) - \frac{1}{4} - \frac{\mu_{2m}}{2m}.
\end{aligned}$$

Hence

$$\lim_{m \rightarrow \infty} I_m = -(c + \frac{1}{2}) \log 2 + \left(\frac{1}{4} + c\right) \log(1 + 2c) - \frac{1}{4} - c. \tag{4.19}$$

From equations (4.17), (4.18) and (4.19) it follows

$$\lim_{m \rightarrow \infty} -\frac{1}{(2m)^2} \log A_{2m} = \frac{3}{4} + \frac{1}{2} \log 2 + \left(\frac{3}{2} + \log 2\right) c + c^2 \log(2c) - \left(c^2 + c + \frac{1}{4}\right) \log(1 + 2c).$$

Second step. when $n = 2m + 1$, we prove by the same method that

$$\lim_{m \rightarrow \infty} -\frac{1}{(2m+1)^2} \log A_{2m+1} = \frac{3}{4} + \frac{1}{2} \log 2 + \left(\frac{3}{2} + \log 2\right) c + c^2 \log(2c) - \left(c^2 + c + \frac{1}{4}\right) \log(1 + 2c).$$

Furthermore it is easy to see that the integral B_n is a particular case of A_n when we take $\mu_n = nc$. Then we have

$$\lim_{n \rightarrow \infty} -\frac{1}{n^2} \log B_n = \lim_{n \rightarrow \infty} -\frac{1}{n^2} \log A_n = E_{2,c}^*.$$

Second case $\beta > 0$. Define for $\nu \in \mathfrak{M}^1(\mathbb{R})$ the energy

$$E_{\beta, \alpha_n}(\nu) = \frac{\beta}{2} \left(\int_{\mathbb{R}^2} \log \frac{1}{|s-t|} \nu(ds) \nu(dt) + \int_{\mathbb{R}} Q_{\beta, \alpha_n}(t) \nu(dt) \right),$$

where

$$Q_{\beta, \alpha_n}(t) = \left(\sqrt{\frac{2}{\beta}} t \right)^2 + \frac{4\alpha_n}{\beta} \log \frac{1}{|t|}.$$

Since

$$Q_{\beta, \alpha_n}(t) = Q_{2, \alpha_n} \circ h(t) + \frac{4\alpha_n}{\beta} \log \sqrt{\frac{2}{\beta}},$$

where $h(t) = \sqrt{\frac{2}{\beta}}t$. Then by proposition 3.3, we obtains

$$E_{\beta, \alpha_n} = \frac{\beta}{2} E_{2, \frac{2\alpha_n}{\beta}} + \frac{\beta}{2} \log \sqrt{\frac{2}{\beta}} + 2\alpha_n \log \sqrt{\frac{2}{\beta}}.$$

We saw from lemma 3.5 that

$$\lim_{n \rightarrow \infty} E_{2, \frac{2\alpha_n}{\beta}} = E_{2, \frac{2c}{\beta}}^* \text{ and } \lim_{n \rightarrow \infty} E_{\beta, \alpha_n} = E_{\beta, c}^*.$$

From the first case $\beta = 2$ and simple computation we deduce the desired result. ■

5 Proof of theorem 4.1

Recall the statistical distribution ν_n is defined by: for all bounded continuous function f on \mathbb{R} ,

$$\int_{\mathbb{R}} f(t) \nu_n(dt) = \mathbb{E}_{n, \mu_n} \left(\frac{1}{n} \sum_{i=1}^n f(\lambda_i) \right),$$

where \mathbb{E}_{n, μ_n} is the expectation with respect the probability on \mathbb{R}^n

$$\mathbb{P}_{n, \mu_n}(d\lambda) = \frac{1}{Z_n} e^{-n \sum_{i=1}^n \lambda_i^2} \prod_{i=1}^n |\lambda_i|^{2\mu_n} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta d\lambda_1 \cdots d\lambda_n.$$

Let Define on \mathbb{R}^n the function :

$$K_n(x) = \frac{\beta}{2} \sum_{i \neq j} \log \frac{1}{|x_i - x_j|} + (n-1) \sum_{i=1}^n Q_{\alpha_n}(x_i),$$

where $Q_{\alpha_n} = x^2 + 2\alpha_n \log \frac{1}{|x|}$ and $\alpha_n = \frac{\mu_n}{n}$.

The probability \mathbb{P}_{n, μ_n} concentrates in a neighborhood of the points where the function $K_n(x)$ attains its infimum:

Proposition 5.1 *Let $\varepsilon > 0$ and $A_{n, \varepsilon} = \{x \in \mathbb{R}^n \mid K_n(x) \leq (E_{\beta, c}^* + \varepsilon)n^2\}$. Then $A_{n, \varepsilon}$ is compact and*

$$\lim_{n \rightarrow \infty} \mathbb{P}_{n, \mu_n}(A_{n, \varepsilon}) = 1.$$

This proposition can be found in [4], lemma IV.5.2. We give the proof.

Proof. Recall that $h_{\alpha_n}(x) = Q_{\alpha_n}(x) - \log(1 + x^2)$. Since h_{α_n} is lower semicontinuous and

$$K_n(x) \geq (n-1) \sum_{i=1}^n h_{\alpha_n}(x_i), \quad \lim_{x_i \rightarrow \pm\infty} h_{\alpha_n}(x_i) = +\infty,$$

then $A_{n,\varepsilon}$ is closed and bounded hence it is compact.

Let $\varepsilon > 0$, from the definition of $A_{n,\varepsilon}$ we have on $\mathbb{R}^n \setminus A_{n,\varepsilon}$

$$K_n(x) > (E_{\beta,c}^* + \varepsilon)n^2,$$

then

$$\mathbb{P}_{n,\mu_n}(\mathbb{R}^n \setminus A_{n,\varepsilon}) \leq \frac{1}{Z_n} e^{-(E_{\beta,c}^* + \varepsilon)n^2} \left(\int_{\mathbb{R}} e^{-Q_{\alpha_n}(x)} dx \right)^n.$$

Furthermore

$$\int_{\mathbb{R}} e^{-Q_{\alpha_n}(x)} dx = \int_{\mathbb{R}} |x|^{2\alpha_n} e^{-x^2} dx = \Gamma(\alpha_n + \frac{1}{2}).$$

By continuity of the gamma function we have $\lim_{n \rightarrow \infty} \Gamma(\alpha_n + \frac{1}{2}) = \Gamma(c + \frac{1}{2})$. Since from proposition 4.6 $\lim_{n \rightarrow \infty} -\frac{1}{n^2} \log Z_n = E_{\beta,c}^*$. Then there is n_0 such that for all $n \geq n_0$,

$$\frac{1}{Z_n} \leq e^{(E_{\beta,c}^* + \frac{\varepsilon}{2})n^2}.$$

Using all those arguments we obtain for n large enough

$$\mathbb{P}_{n,\mu_n}(\mathbb{R}^n \setminus A_{n,\varepsilon}) \leq \left(\Gamma(c + \frac{1}{2}) + \varepsilon \right)^n e^{-\frac{\varepsilon}{2}n^2}.$$

Which complete the proof. ■

Proof of theorem 4.1. We keep those notations:

$$k_{\alpha_n}(s, t) = \log \frac{1}{|s-t|} + \frac{1}{2} Q_{\alpha_n}(s) + \frac{1}{2} Q_{\alpha_n}(t),$$

for $\ell > 0$,

$$k_{\alpha_n}^\ell(s, t) = \inf(k_\delta(s, t), \ell).$$

$$h_{\alpha_n}(t) = Q_{\alpha_n}(t) - \log(1 + t^2),$$

and $h(t) = \inf(h_{a_1}(t), h_{a_2}(t))$, h_{a_1} and h_{a_2} are the functions used on the proof of proposition 3.6 where $a_1, a_2 \geq 0$

For a bounded continuous function f on \mathbb{R} , defined on \mathbb{R}^n the continuous function

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n f(x_i),$$

Let $\varepsilon > 0$, the set $A_{n,\varepsilon}$ is compact, hence F_n attains its supremum at some point in $A_{n,\varepsilon}$ say

$$x_\varepsilon^{(n)} = (x_{1,\varepsilon}^{(n)}, \dots, x_{n,\varepsilon}^{(n)}).$$

We obtain

$$\int_{\mathbb{R}} f(t) \nu_n(dt) \leq F_n(x_\varepsilon^{(n)}) + \|f\|_\infty (1 - \mathbb{P}_{n,\mu_n}(A_{n,\varepsilon})).$$

To the point $x_\varepsilon^{(n)}$ we associate the probability measure on \mathbb{R}

$$\sigma_{n,\varepsilon} = \frac{1}{n} \sum_{i=1}^n \delta_{x_{i,\varepsilon}^{(n)}}.$$

The previous inequality can be written

$$\int_{\mathbb{R}} f(t) \nu_n(dt) \leq \int_{\mathbb{R}} f(t) \sigma_{n,\varepsilon}(dt) + \|f\|_\infty (1 - \mathbb{P}_{n,\mu_n}(A_{n,\varepsilon})),$$

The truncated energy E^ℓ of the measure $\sigma_{n,\varepsilon}$ satisfies:

$$E^\ell(\sigma_{n,\varepsilon}) \leq \frac{\ell}{n} + (E_{\beta,c}^* + \varepsilon).$$

From the inequality

$$(n-1) \sum_{i=1}^n h(x_i) \leq K_n(x),$$

we obtain

$$\int_{\mathbb{R}} h(t) \sigma_{n,\varepsilon}(dt) \leq \frac{n}{n-1} (E_{\beta,c}^* + \varepsilon).$$

This implies that the sequence $\sigma_{n,\varepsilon}$ is relatively compact for the weak topology. There is a sequence n_j going to ∞ such that the subsequence $\sigma_{n_j,\varepsilon}$ converges in the weak topology:

$$\lim_{n \rightarrow \infty} \sigma_{n_j,\varepsilon} = \sigma_\varepsilon.$$

We may also assume in the weak topology that

$$\lim_{j \rightarrow \infty} \nu_{n_j} = \limsup_n \nu_n.$$

the limit measure satisfies

$$E^\ell(\sigma_\varepsilon) \leq E_{\beta,c}^* + \varepsilon.$$

The rest of the proof is analogous to the proof of theorem IV.5.1 [4].

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